On the development of choice models based on quantum theory

Stephane Hess
Universidad de Chile, 24 June 2019
In memory of John Polak
Recent developments on behavioural realism
First insights from mathematical psychology
Quantum Probability
Quantum rotation
QPM for “moral” decisions
Adaptations for dynamic data
Recent developments on behavioural realism
Trends in choice modelling

- Dramatic increase in complexity of models in last two decades
- Increased focus on of heterogeneity
- And departures from “rational” models, i.e. random utility maximisation (RUM)
Big influence of work in behavioural economics

Allowing for heterogeneous decision rules in discrete choice models: an approach and four case studies
Stephane Hess · Amanda Stathopoulos · Andrew Daly
RUM has important benefits: let’s not throw the baby out with the bathwater!

• RUM may not represent the choice process used by decision makers but:
  • results directly usable for welfare analysis
  • seems to perform well for forecasting
  • decades of good experiences
• Three key questions:
  • how far can we go within RUM framework?
  • does explicit modelling of behavioural process matter more than retention of microeconomic framework?
  • where should we turn to if we are willing to let go of the advantages of RUM?
Behavioural “anomalies”: categorisation by Hess, Daly & Batley (2018)

**Generally consistent**
- Anchoring
- Status quo bias
- Zero cost bias
- Mental accounting
- Elimination by aspects

**Not consistent in general or in some cases**
- Lexicography
- Decoy, context, framing
- Simplification
- Regret
- Reference dependence
RUM vs RRM

• Utility: \( V_{int} = \sum_k \beta_k x_{intk} \)
• Regret: \( R_{int} = \sum_{j \neq i} \sum_k \ln(1 + \exp[\beta_k \cdot (x_{jntk} - x_{intk})]) \)
• Both models assume that errors follow a type I EV distribution

\[
\begin{align*}
P_{int,U} &= \frac{\exp(V_{int})}{\sum_{j=1..J} \exp(V_{jnt})} \\
P_{int,R} &= \frac{\exp(-R_{int})}{\sum_{j=1..J} \exp(-R_{jnt})}
\end{align*}
\]

• Random regret minimisation is a logit model, just not a RUM consistent logit model
What if we want to move away from RUM?

• Need to accept loss of key RUM benefits, notably in welfare analysis
• Potentially substantial increases in complexity and estimation issues
• Many existing comparisons (e.g. RUM vs RRM) have shown only small differences in performance
• Likely a result of the models being too “similar”
• We instead look at mathematical psychology
First insights from mathematical psychology
Models from mathematical psychology

• Very active field of research
• Many similarities with choice modelling
• But they speak a different language!
• Almost no real world applications
• Major focus on short-term dynamic preference evolution
• We have done a lot of work with Decision Field Theory (DFT)
Preference accumulation models

- Dynamic models of preference creation
- Consider different attributes of the alternatives at different points in time

<table>
<thead>
<tr>
<th>Cost</th>
<th>Train</th>
<th>Bus</th>
<th>Car</th>
</tr>
</thead>
<tbody>
<tr>
<td>High</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Low</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Average</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Time</th>
<th>Low</th>
<th>High</th>
<th>Average</th>
</tr>
</thead>
</table>

<table>
<thead>
<tr>
<th>Environment</th>
<th>Average</th>
<th>Good</th>
<th>Poor</th>
</tr>
</thead>
</table>
Basic DFT equations

\[ P_t = S \cdot P_{t-1} + V_t \]

\[ V_t = C \cdot M \cdot W_t + \varepsilon_t \]

- Preference vector \( P_t \) at a given timestep \( t \), updates over time
  - Preference vector ≠ probability
- \( S \) = feedback matrix
- \( P_0 \) = initial preference vector
- \( V_t \) = valence vector (how much preferences update at \( t \))
- \( M \) = attribute matrix
- \( C \) = contrast matrix (to centre the values around zero)
- \( W_t \) = weights vector, with one attribute considered in each evaluation
- \( \varepsilon_t \) = \( \mathcal{N}(0,\sigma) \) error term
DFT probabilities

• $P_t$ converges to a multivariate normal distribution, e.g. with 3 alternatives:

\[
Pr[P_t[A] - P_t[B] > 0 \cap P_t[A] - P_t[C] > 0] = \int_{X \succeq 0} \exp\left[-(X - \Gamma)'\Lambda^{-1}(X - \Gamma)/2\right]/(2\pi |\Lambda|^{0.5})dX
\]

with $X = [P_t[A] - P_t[B], P_t[A] - P_t[C]]'$, $\Gamma = L\xi_t$, $\Lambda = L\Omega_tL'$ and

\[
L = \begin{bmatrix}
1 & -1 & 0 \\
1 & 0 & -1
\end{bmatrix}
\]

• Need mean and covariance

\[
E[P_t] = \xi_t = \sum_{k=0}^{t-1} S^k \cdot \mu + S^t \cdot P_0
\]

\[
Cov[P_t] = \Omega_t = Cov\left[\sum_{k=0}^{t-1} S^k \cdot V_{t-k} + S^t \cdot P_0\right]
\]

\[
= (I - S)^{-1} (I - S^t) \cdot \mu + S^t \cdot P_0
\]

• The summation in the covariance matrix historically made DFT computationally expensive until Hancock et al. (2018)
DFT probabilities

Previously was only ever calculated at (t=∞), when values stabilised.

New method allows for influence of initial preferences.

<table>
<thead>
<tr>
<th>Alternative</th>
<th>P(t=10)</th>
<th>P(t=100)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Train</td>
<td>0.184</td>
<td>0.086</td>
</tr>
<tr>
<td>Car</td>
<td>0.499</td>
<td>0.469</td>
</tr>
<tr>
<td>Bus</td>
<td>0.261</td>
<td>0.428</td>
</tr>
<tr>
<td>Bike</td>
<td>0.056</td>
<td>0.016</td>
</tr>
</tbody>
</table>
Danish value of time dataset

• 2 alternatives described by cost and time:
  • MNL : LL = -2,301.53
  • Non-linear MNL : LL = -2,212.10
  • DFT : LL = -2,015.35
UK commuter dataset

• 3 alternatives, described by cost, time, rate of delays, average length of delays, crowding and provision of a delay information service:

• MNL : $LL = -3,391.79$
• RRM : $LL = -3,379.96$
• DFT : $LL = -3,346.23$
Swiss value of time survey

• MNL: LL = -1,667.97
• DFT: LL = -1,595.85

• Can also do a DFT with random parameters: LL = -1430.41
RP data

Results from UK value of travel time study

• MNL: -370.05
• RRM: -373.31
• DFT: -363.31
Problems with DFT

• Very exciting results, especially on dynamic data
• But computationally very demanding
• True benefits will also likely only arise if we capture additional process information
Quantum mechanics in early 1900s

- **Sequence** that measurements for physical variables are taken in affects final observed values

- Heisenberg uncertainty principle:
  - cannot precisely measure momentum and position of a particle at the same time

- Total uncertainty: \( \sigma_x \cdot \sigma_\rho \geq \frac{h}{4\pi} \)
  - \( \sigma_x \) and \( \sigma_\rho \) sd of position and momentum
  - \( h \) is the Planck constant

- Three possible propositional variables:

<table>
<thead>
<tr>
<th>( A ): momentum in interval ([\rho_1, \rho_2])</th>
<th>( B ): particle is in interval ([x_1, x_2])</th>
<th>( C ): particle is in interval ([x_2, x_3])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Imagine a situation where:</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( A \cap (B \cup C) = \text{TRUE} \ (\sigma_x \cdot \sigma_\rho \geq \frac{h}{4\pi}) )</td>
<td>( A \cap B = \text{FALSE} \ (\sigma_x \cdot \sigma_\rho &lt; \frac{h}{4\pi}) )</td>
<td>( A \cap C = \text{FALSE} \ (\sigma_x \cdot \sigma_\rho &lt; \frac{h}{4\pi}) )</td>
</tr>
</tbody>
</table>

- Distributivity law of classical probability no longer holds \( \Rightarrow \) breakdown of classical logic
  - \( A \cap (B \cup C) = \text{TRUE} \) but \((A \cap B) \cup (A \cap C) = \text{FALSE}\)
Quantum probability

- Work in physics led to development of Quantum Probability/Quantum Logic
- Gradually being introduced into psychology as 'Quantum cognition'
- Controls for/predicts order effects
  - E.g. destination then mode?
- Also used to test the impact of nudges
- Our work has developed a choice model based on quantum probability
A simple binary example (car vs train)

- Alternatives represented by $|T\rangle$ and $|C\rangle$
- Decision-maker has some initial state, $|z\rangle$
- The action of making a choice (or judgement) results in a “change of state”
- Move from initial state vector and “project” onto vector for chosen alt.
- $\rho_T$ represents scalar projection of $|z\rangle$ onto a straight line parallel to $|T\rangle$
- Length of projection denoted $|\rho_T|$ 
- Longer projection for an alternative means higher probability of being chosen
Why use quantum logic?

• Behavioural state initially `indefinite' and often created rather than just recorded by an attempt to measure it
  • Ask for choices before attitudes, or other way around?
  • We cannot measure both without influence/bias
• Psychologists have put forward argument that cognition behaves like a wave
  • Preference exists as some measurable definite state only when decision-maker makes up their mind
• One of the most crucial areas for quantum concepts, however, is the idea of interferences or nudges
Impact of a nudge/reminder

- Question before choice: “are you environmentally friendly?”
- Answer “I am env. friendly”
  - state moves from initial starting state
  - projected onto “env. friendly” vector
- Now projections from there onto train and car vectors
  - Probability of train increases
Example with three alternatives

- Choice scenario X represented geometrically as a subspace $L_x$ in a J-dimensional Hilbert space.
- Number of discrete “events”, represented by orthonormal vectors: $|x_1\rangle$, $|x_2\rangle$, ...
- With J=3, the lengths of the three projections can be visualised as the three sides of the cuboid in 3-dimensional space.
- State vector has length 1 and cuts diagonally from one corner to opposite corner of cuboid.
  - Pythagoras: the squared lengths of the projections must sum to one
    $$\sum_{j=1}^{J} |\rho_{x_j}|^2 = 1$$
Operationalisation

• Basic idea is for user to define the projections
  • as with utilities in a random utility model
• If the sum of squares of these equals 1, then the squared projections are probabilities
• If not, then we need a further normalisation
  • Instead of $P_i = \rho_i^2$, we have $P_i = \frac{\rho_i^2}{\sum_{j=1}^{J} \rho_j^2}$
• Easy to satisfy constraint for binary choices
Binary example

\[
\cos \theta = \frac{\text{adj}}{\text{hyp}} = \frac{|\rho_{x_1}|}{1}
\]

\[
\sin \theta = \frac{\text{opp}}{\text{hyp}} = \frac{|\rho_{x_2}|}{1}
\]

\[
\cos^2 \theta + \sin^2 \theta = 1
\]

\[
|\rho_{x_1}|^2 + |\rho_{x_2}|^2 = 1
\]

\[
P(Alt1) = \cos^2 \theta
\]

\[
P(Alt2) = \sin^2 \theta
\]
Imagine that alternative 2 becomes more attractive

From scenario 1 to 2:

\[
\theta \uparrow \\
|\rho_{x_1}| \downarrow \& |\rho_{x_2}| \uparrow \\
\cos \theta \downarrow \& \sin \theta \uparrow \\
P(Alt1) \downarrow \& P(Alt2) \uparrow
\]
Regret based approach for 3+ alternatives

- An alternative possibility for the projections is to use regret functions from random regret minimisation (RRM).
- Potential to work for a QPM as the logarithm guarantees that only positive values are generated from the pairwise comparisons between alternatives.
- ‘Quantum pairwise comparison version A (QPCA)‘:

\[
|\rho_{int}| = \delta_{QPCA,i} + I_0 + \sum_{k=1}^{K} \sum_{j \neq i} w_{t_{ij}} \cdot \ln\left(1 + e^{\beta_k (x_{jntk} - x_{intk})}\right)
\]

where \(\delta\) are alternative-specific constants and \(I_0\) is a constant that has the same value across all alternatives.

- \(P_i = \frac{\rho_i^2}{\sum_{j=1}^{J} \rho_j^2}\)
LBA based approach for 3+ alternatives

• Use of drift rate functions from the multi-attribute linear ballistic accumulator model

\[ |\rho_{int}| = \delta_{QPCB,i} + I_0 + \sum_{j \neq i} \sum_{k=1}^{K} w_{t_{i,j}} \cdot (w_{x_{k,i,j}} \cdot \beta_k \cdot (x_{k,j} - x_{k,i})) \]

• A natural way to incorporate asymmetries in attribute differences (where regret only looks at loss)

\[ w_{x_{k,i,j}} = -\exp \left( (\lambda_1 \cdot [x_{k,i} \geq x_{k,j}] + \lambda_2 [x_{k,i} < x_{k,j}] \cdot \beta_k \cdot |x_{k,i} - x_{k,j}|) \right) \]
## Model fit (BIC) on static RP and SP datasets

<table>
<thead>
<tr>
<th>type</th>
<th>Danish route choice</th>
<th>UK mode choice</th>
<th>Swiss route choice</th>
<th>UK rail company choice</th>
</tr>
</thead>
<tbody>
<tr>
<td>alternatives</td>
<td>SP</td>
<td>SP</td>
<td>SP</td>
<td>RP</td>
</tr>
<tr>
<td>MNL</td>
<td>4627.54</td>
<td>6802.97</td>
<td>3376.74</td>
<td>812.54</td>
</tr>
<tr>
<td>RRM</td>
<td>4627.54</td>
<td>6809.93</td>
<td>3376.74</td>
<td>819.07</td>
</tr>
<tr>
<td>DFT</td>
<td>4062.5</td>
<td>6743.26</td>
<td>3252</td>
<td>805.66</td>
</tr>
<tr>
<td>QPM</td>
<td>4041.19</td>
<td>6753.78</td>
<td>3187.88</td>
<td>812.46</td>
</tr>
</tbody>
</table>

- **BIC relative to worst model for given data**
  - Danish VTT route choice
  - UK commuter mode choice
  - Swiss VTT route choice
  - UK rail operator

- **Models**
  - MNL
  - RRM
  - DFT
  - QPM

- **Values**
  - 0.8
  - 0.82
  - 0.84
  - 0.86
  - 0.88
  - 0.9
  - 0.92
  - 0.94
  - 0.96
  - 0.98
  - 1
  - 1.02

- **Institute for Transport Studies**
  - UNIVERSITY OF LEEDS

- **Choice Modelling Centre**
Quantum rotation
Imagine a situation where the attributes of the alternatives remain the same, but something else changes.

- E.g. the context, the sequence of choices, etc.
- We cover this by introducing the notion of quantum rotation.
Implementation

- Group together original projections:
  \[ |\rho| = |\rho_1, ..., \rho_J| \]
- Matrix multiplication on projections
- Projections after rotation:
  \[ |\rho^*| = \begin{bmatrix} m_{11} & \cdots & m_{1J} \\ \vdots & \ddots & \vdots \\ m_{J1} & \cdots & m_{JJ} \end{bmatrix} |\rho| \]
  with the normalisation that \( m_{11} = 1 \)
Illustration of quantum rotation on time-money trade-off

<table>
<thead>
<tr>
<th></th>
<th>Option A</th>
<th>Option B</th>
</tr>
</thead>
<tbody>
<tr>
<td>One way fuel cost</td>
<td>£33.30</td>
<td>£35.00</td>
</tr>
<tr>
<td>One way travel time by car</td>
<td>4 hours 23 minutes</td>
<td>3 hours 30 minutes</td>
</tr>
</tbody>
</table>

- Subjective valuations: fast-expensive is better in scenario 1, and cheap-slow in scenario 2
- Moving to right $m_{\text{cheap}} = \begin{bmatrix} 1 & -0.053 \\ 0.102 & 1.225 \end{bmatrix}$
- Moving cost to top $m_{\text{cost}} = \begin{bmatrix} 1 & 0.043 \\ -0.008 & 0.898 \end{bmatrix}$
- Shows that 3-5 is not the reverse of 5-3
- Alternatives do better on left, and cost matters more if shown first

<table>
<thead>
<tr>
<th></th>
<th>Scenario 1</th>
<th>Scenario 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length of projection</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cheap, slow</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>Fast, expensive</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>Cheaper alternative first, travel time first</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cheap, slow</td>
<td>26.5%</td>
<td>73.5%</td>
</tr>
<tr>
<td>Fast, expensive</td>
<td>73.5%</td>
<td>26.5%</td>
</tr>
<tr>
<td>Cheaper alternative second, travel time first</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cheap, slow</td>
<td>15.3%</td>
<td>57.2%</td>
</tr>
<tr>
<td>Fast, expensive</td>
<td>84.7%</td>
<td>42.8%</td>
</tr>
<tr>
<td>Cheaper alternative second, travel cost first</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Cheap, slow</td>
<td>34.1%</td>
<td>78.9%</td>
</tr>
<tr>
<td>Fast, expensive</td>
<td>65.9%</td>
<td>21.1%</td>
</tr>
</tbody>
</table>
QPM for “moral” decisions
“Moral” choices

- Choices where a decision maker feels that the alternatives can to some extent be categorised as “right” or “wrong”
How about modelling “moral” choices?

• Much attention in economics and psychology, but rarely considered in choice modelling literature

• Moral preferences difficult to investigate outside of the laboratory, with experimental methods often suffering from low external validity

• However, moral choice behaviour has become more prominent to the travel behaviour modelling community through, for example, the reinvention of the trolley problem as a self-driving car problem

• Significant interest in modelling moral choices generated by Chorus’s group in Delft, looking at “taboo” trade-offs

• We test whether quantum rotations can be used to accurately capture changes in choice context within moral choice scenarios
Dataset 1

• Chorus’s “taboo trade-offs” dataset
• Choice between the introduction of a new transport policy or keeping the status quo
• Each new policy offered simply an increase or decrease for four attributes:
  • 300 EUR vehicle ownership tax
  • 20 minutes travel time for each car commuter per day
  • 100 serious injuries in traffic accidents
  • 5 deaths in traffic accidents
• Results in 16 possible new policies
• 99 decision-makers, 1,584 choices
• Some choice tasks involve a `taboo trade-off':
  • decreasing tax or travel time (a secular attribute) at the cost of increasing the number of injuries or deaths (a sacred attribute)
QPM results

• Chorus et al. (2018): “Generic Taboo Trade-Off Aversion” (TTOA) model: LL=-719.5

• Simple QPM model and three models with rotations

• All three have a value of greater than 1 for element $m_{2,2}$, which, all else being equal, would mean that the decision-maker is more likely to choose the SQ alternative in the presence of a taboo trade-off

• However, different estimates for $m_{1,2}$ and $m_{2,1}$ mean that the overall impact is more complex

<table>
<thead>
<tr>
<th>TQ</th>
<th>Parameters</th>
<th>Log-likelihood</th>
<th>Quantum rotation matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>$M_{1,1}$</td>
</tr>
<tr>
<td>Basic</td>
<td>5</td>
<td>-722.94</td>
<td>1.00</td>
</tr>
<tr>
<td>QR1</td>
<td>6</td>
<td>-717.33</td>
<td>1.00</td>
</tr>
<tr>
<td>QR2</td>
<td>7</td>
<td>-714.21</td>
<td>1.00</td>
</tr>
<tr>
<td>QR3</td>
<td>8</td>
<td>-713.63</td>
<td>1.00</td>
</tr>
</tbody>
</table>
## Predicted support for new policy

<table>
<thead>
<tr>
<th>Scenario</th>
<th>Attributes</th>
<th>Taboo Trade-Off?</th>
<th>Share of support</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Tax</td>
<td>Time</td>
<td>Injuries</td>
</tr>
<tr>
<td>1</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>4</td>
<td>-</td>
<td>+</td>
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<tr>
<td>5</td>
<td>+</td>
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<td>10</td>
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<tr>
<td>11</td>
<td>-</td>
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<td>+</td>
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<td>14</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>15</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
<tr>
<td>16</td>
<td>+</td>
<td>-</td>
<td>+</td>
</tr>
</tbody>
</table>

Mean absolute deviation from true share of support (percentage points; all choice tasks): 3.03
Mean absolute deviation from true share of support (percentage points; taboo tasks only): 2.68
Dataset 2

• Decision-makers completing two sets of choice tasks based on an individual's willingness to accept longer commutes for better salaries

• First set involved trade-offs between the individual's current travel time and salary or an increased salary (of 500 or 1000 SEK in net wage per month) at a cost of an increase in one-way travel time (of either 10 or 25 minutes)

• Second set additionally included attributes for increased travel time and salaries for the partner of the decision-maker, meaning that the decision-maker has to make choices about who to prioritise

• All choice tasks included a status quo alternative, a new alternative and a `I am indifferent' option

• 1,179 households, 2,358 individuals, 20,041 choice observations.
QPM vs RRM

- Run models with joint parameters for two types of scenarios and models with separate parameters
- QPM outperforms RRM with either specification

<table>
<thead>
<tr>
<th>Model</th>
<th>Separate Parameters</th>
<th>Free Parameters</th>
<th>Log-likelihood</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>RRM</td>
<td>No</td>
<td>6</td>
<td>-12,784.21</td>
<td>25,628</td>
</tr>
<tr>
<td>RRM</td>
<td>Yes</td>
<td>10</td>
<td>-12,426.71</td>
<td>24,952</td>
</tr>
<tr>
<td>QPCA</td>
<td>No</td>
<td>7</td>
<td>-12,624.13</td>
<td>25,318</td>
</tr>
<tr>
<td>QPCA</td>
<td>Yes</td>
<td>12</td>
<td>-12,289.38</td>
<td>24,698</td>
</tr>
</tbody>
</table>
Use quantum rotation instead of separate params

- Inconsistency or “change of mindset” incurred through changing from thinking about just yourself compared to yourself and your partner could be captured by a quantum rotation
- We again test diagonal, symmetric and fully flexible rotation matrices
- 3 alternatives, so up to 8 estimated parameters in rotation matrix

<table>
<thead>
<tr>
<th>#</th>
<th>Model</th>
<th>Parameters</th>
<th>Log-likelihood</th>
<th>Log-likelihood Gain</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Over basic model</td>
</tr>
<tr>
<td>1</td>
<td>Basic</td>
<td>7</td>
<td>-12,624.13</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>Separate</td>
<td>12</td>
<td>-12,289.38</td>
<td>334.75</td>
</tr>
<tr>
<td>3</td>
<td>QR1</td>
<td>9</td>
<td>-12,436.03</td>
<td>188.10</td>
</tr>
<tr>
<td>4</td>
<td>QR2</td>
<td>12</td>
<td>-12,332.37</td>
<td>291.76</td>
</tr>
<tr>
<td>5</td>
<td>QR3</td>
<td>15</td>
<td>-12,278.06</td>
<td>346.07</td>
</tr>
</tbody>
</table>
Results for rotation

• Alternatives in matrix:
  • 1: status quo
  • 2: increase travel times and salaries
  • 3: indifferent option

• High values for $m_{1,3}$ and $m_{2,3}$
  • if an individual is indifferent for a single person choice scenario, they will likely not still be indifferent if there are additionally changes for their partner

\[
M = \begin{bmatrix}
1.000 & -0.462 & 4.839 \\
-0.045 & 0.116 & 2.638 \\
0.151 & -0.067 & -0.646 \\
\end{bmatrix}
\]
Adaptations for dynamic data
Data: US 101 (Hollywood freeway)

- 640 metre cross-section of road
- 5 lanes as well as an auxiliary lane which connects an on-ramp and off-ramp
- Data collected using cameras over 8 segments
Data: US 101 (Hollywood freeway)

- Lane merging behaviour by drivers who join US-101 from Ventura on-ramp
- Factors impacting decision on when to merge are constantly changing
- 45-minute video, 399 vehicles starting on on-ramp and merging onto lane 5
- Use data for every 0.1 seconds
- For simplicity, assume that drivers take a second to react to visual stimuli
Key considerations for dynamic QPM

- Basic setting: angle/projections only influenced by current choice settings
- Memory effect: allow impact of recent attribute values, with decreasing importance
- Changing state of mind: allow for change in context through quantum rotation
Earlier, we introduced quantum rotation. Now we look at using this in a situation where the attributes of the alternatives remain the same, but the “goal posts” are moved. For example, distances between cars remain the same, but you’re getting closer to the junction. Instead of rotating axes, we rotate the angle of the state vector.
\[ \theta = \text{dependent on traffic conditions} \]

**Initial projections:**

\[ |\rho_{x_1}| = \cos \theta \quad \& \quad |\rho_{x_2}| = \sin \theta, \]

\[ v = \text{dependent on location on motorway} \]

**Rotation:**

\[ M = \begin{pmatrix} 1 & v \cdot r_{12} \\ v \cdot r_{21} & 1 + v \cdot r_{22} \end{pmatrix} \]

Then as before:

\[ \cos^2 \theta + \sin^2 \theta = 1 \]

\[ P(\text{Alt1}) = \cos^2 \theta \]

\[ P(\text{Alt2}) = \sin^2 \theta \]
Results

• Can compare QPM and DFT against an equivalent random utility model (RUM)
  • DFT naturally incorporates decay/memory. QPM and RUM can instead use an exponential decay function to capture impact of ‘past’ attributes

• Model results:

<table>
<thead>
<tr>
<th></th>
<th>LL</th>
<th>pars</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>DFT</strong></td>
<td>-1750.80</td>
<td>12</td>
<td>3625.79</td>
</tr>
<tr>
<td><strong>QPM</strong></td>
<td>-1756.04</td>
<td>14</td>
<td>3656.98</td>
</tr>
<tr>
<td><strong>RUM</strong></td>
<td>-1768.58</td>
<td>12</td>
<td>3661.36</td>
</tr>
</tbody>
</table>
### Traffic variables: impact of lane

<table>
<thead>
<tr>
<th>Model</th>
<th>Impact on model</th>
<th>On-ramp</th>
<th>Lane 6</th>
<th>Off-ramp</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFT</td>
<td>On valence for stay</td>
<td>0</td>
<td>-0.027</td>
<td>-0.055</td>
</tr>
<tr>
<td>QPM</td>
<td>Impacts quantum rotation</td>
<td>0</td>
<td>-0.85</td>
<td>-10.45</td>
</tr>
<tr>
<td>RUM</td>
<td>On utility to stay</td>
<td>0</td>
<td>-1.05</td>
<td>-1.98</td>
</tr>
</tbody>
</table>

- Drivers increasingly more likely to merge as they reach end of the ramp
- For DFT and RUM, effect is doubled as between lane 6 and Lankershim off-ramp
- Quantum model sees a far more significant change
  - a much larger ‘change in perspective’
Traffic variables: impact of gap size

- Very rapid damping of importance of gap size, a bit less so for QPM
- Gap behind matters more than gap in front

<table>
<thead>
<tr>
<th>Model</th>
<th>Gap impacts</th>
<th>Gap in front</th>
<th>Gap behind</th>
<th>Relative importance of front gap to behind gap</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFT</td>
<td>Valence for merging</td>
<td>0.9981</td>
<td>0.9921</td>
<td>0.7891</td>
</tr>
<tr>
<td>QPM</td>
<td>Angle for merging</td>
<td>0.9848</td>
<td>0.9575</td>
<td>0.8667</td>
</tr>
<tr>
<td>RUM</td>
<td>Utility for merging</td>
<td>0.9989</td>
<td>0.9905</td>
<td>0.7522</td>
</tr>
</tbody>
</table>

![Graph showing time headway vs relative scale]
Decay/memory parameters

- Multiplication factor ($\alpha_d$) applied to traffic utilities/preference values/projection lengths from previous 1/10 sec

<table>
<thead>
<tr>
<th>Memory/decay parameters ($\alpha_d$)</th>
<th>Exponential decay (i.e.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>DFT</td>
<td>0.808</td>
</tr>
<tr>
<td>QPM</td>
<td>0.576</td>
</tr>
<tr>
<td>RUM</td>
<td>0.776</td>
</tr>
</tbody>
</table>

- QPM again more different from other models
Conclusions & next steps

- DFT and QPM present much bigger departures from standard models than commonly tried
- Great promise in terms of behavioural flexibility
- DFT is costly in estimation, QPM is no worse than “standard” models
- Some of the computational cost of DFT goes away in application, so implementation in simulation tools is more feasible
Thank you!